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ORIGINAL PAPER

# Effect of road pricing on the spatial distribution of traffic flow in a grid network

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## Abstract

This paper develops a continuous approximation model for analyzing the effect of road pricing on the spatial distribution of traffic flow. The traffic flow density, which describes traffic flow as a function of position, is derived for a rectangular city with a grid network. The analytical expression for the traffic flow density demonstrates how the toll level, the size and shape of the toll area, and the shape of the city affect the spatial distribution of traffic flow. As the size of the toll area increases, reducing the traffic flow density at the city center becomes difficult. As the aspect ratio of the toll area increases, the traffic flow density at the city center increases. The shape of the city has less impact on the traffic flow density than the shape of the toll area.

**Keywords** Transportation · Traffic flow density · Rectilinear distance · Continuous approximation

**Mathematics Subject Classification** 90B06 · 90B85

## 1 Introduction

Road pricing, which is regarded as one of the most effective means to reduce traffic congestion, has been introduced in several cities such as London, Singapore, and Stockholm. A toll is imposed on vehicles per crossing of the cordon line surrounding a toll area (cordon-based pricing), per distance of travel (distance-based pricing), per time of travel (time-based pricing), or per day in a toll area (area-based pricing). Road pricing encourages travelers to adjust the number of trips and the travel route, thus affecting the spatial distribution of traffic flow. Examining how road pricing affects the

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spatial distribution of traffic flow provides a fundamental understanding of the effect of road pricing.

Two major approaches for addressing road pricing are based on discrete network models and continuous approximation models. The former rely on detailed traffic data and focus on developing efficient algorithms for large-scale problems, whereas the latter rely on approximated functions of travel demand and focus on finding fundamental relationships between variables. By providing analytical solutions, the continuous approximation models can help reveal managerial insights and supplement the discrete network models. The discrete network models on road pricing have studied cordon-based pricing [1–6] and area-based pricing [7–13] on actual networks. May and Milne [14] compared cordon-based, distance-based, time-based, and delay-based pricing schemes. The continuous approximation models have studied cordon-based and area-based pricing in idealized structures such as a linear city [15–19], a circular city [20], a grid network [21], and a radial-arc network [22, 23].

In this paper, we present a continuous approximation model for analyzing the effect of road pricing on the spatial distribution of traffic flow. The continuous approximation yields an analytical expression for the spatial distribution of traffic flow. The analytical expression leads to a better understanding of how the toll level, the size and shape of the toll area, and the shape of the city affect the spatial distribution of traffic flow. The model is thus useful for determining the size and shape of the toll area and the toll level. The model assumes a grid network, which can be found in Kyoto, Beijing, and many cities in North America. Miyagawa [21] dealt with the effect of road pricing in a grid network on the total amount of traffic flow inside the toll area. The effect of road pricing, however, varies according to the location. We then extend the scope to the spatial distribution of traffic flow, which describes traffic flow as a function of position. This extension allows us to examine the locational variation of traffic flow and identify potential congestion areas.

To describe the spatial distribution of traffic flow, the concept of traffic flow density was introduced by Smeed [24] and Holroyd [25]. The traffic flow density expresses the amount of traffic flow passing through a point as a density function. The traffic flow density has been derived for a circular city with the Euclidean distance [26], a circular city with a grid network [27] and a radial-arc network [28], and a square city with a grid network [28]. Subsequent works have considered a sector-shaped city with a radial-arc network [29], a rectangular city with a grid network and a barrier to travel [30], and a circular city with a radial-arc network and road pricing [23]. The time-dependent traffic flow density has also been derived [31, 32]. The traffic flow density in a rectangular city with a grid network and road pricing has not been derived previously.

The remainder of this paper is organized as follows. The next section introduces a grid network model. The following section derives the spatial distribution of traffic flow without road pricing. The penultimate section derives the spatial distribution of traffic flow in road pricing. The final section presents concluding remarks.

## 2 Grid network model

Consider a rectangular city with side lengths $a_1$ and $a_2$, as shown in Fig. 1. Any point in the city is expressed as $(x, y)$ $(0 \leq x \leq a_1, 0 \leq y \leq a_2)$, where the origin of the coordinate system O is at the southwest corner of the city. The city has a dense grid road network. The shortest distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is then given by the rectilinear distance $|x_1 - x_2| + |y_1 - y_2|$. A toll area is represented as a rectangle with side lengths $b_1$ and $b_2$ located at the center of the city $(a_1/2, a_2/2)$. Let $(x_b, y_b)$ be the southwest corner of the toll area, that is, $x_b = (a_1 - b_1)/2$, $y_b = (a_2 - b_2)/2$. The road pricing system in the city is area pricing where all vehicles driving inside the toll area are charged a fixed toll $t$.

Origins and destinations are assumed to be uniformly distributed in the city. The uniform distribution is important as the first approximation and serves as a basis for further analysis with more realistic distributions. The travel cost $C$ for trips of length $R$ is defined as

$$C = \alpha R + t, \tag{1}$$

where $\alpha$ $(> 0)$ is the travel cost per unit distance. Every traveler is assumed to minimize the number of turns in choosing the least cost route. If there exist two least cost routes with an equal number of turns, the amount of traffic flow is equally divided between the two routes. The travel demand $D$ is assumed to decrease with the travel cost $C$ and expressed as

$$D = D_0 \mathrm{e}^{-\beta C}, \tag{2}$$

where $D_0$ is the travel demand when $C = 0$ and $\beta$ $(> 0)$ is a parameter for elasticity. The exponential function has been widely used in spatial interaction models [33].

The traffic flow density is defined as the amount of traffic flow passing through a point in the city. Let $f_x$ and $f_y$ be the densities of traffic flow passing through a point $(x, y)$ in the east–west and north–south directions, respectively. The amount of traffic flow passing through the small segment with length $\mathrm{d}y$ $(\mathrm{d}x)$ in the east–west (north–south) direction is then $f_x \mathrm{d}y$ $(f_y \mathrm{d}x)$. The amount of traffic flow passing through the segment between two points $(x, y_1)$ and $(x, y_2)$ in the east–west direction, denoted by $V_x$, is given by

$$V_x = \int_{y_1}^{y_2} f_x \, \mathrm{d}y, \tag{3}$$

and the amount of traffic flow passing through the segment between two points $(x_1, y)$ and $(x_2, y)$ in the north–south direction, denoted by $V_y$, is given by

$$V_y = \int_{x_1}^{x_2} f_y \, \mathrm{d}x, \tag{4}$$

as shown in Fig. 1.

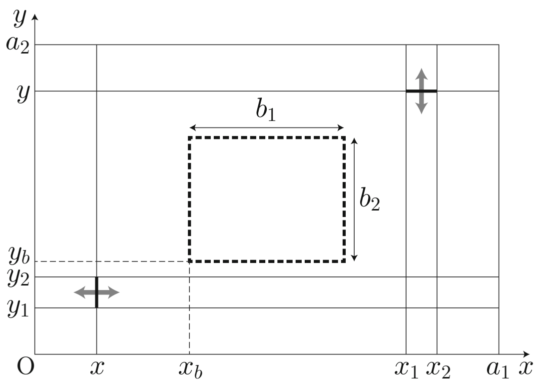


**Fig. 1** Rectangular city with a grid network

## 3 Traffic flow density without road pricing

In this section, we derive the traffic flow density when no toll is charged, that is, $t = 0$. Although Vaughan [28] and Miyagawa [30] derived the traffic flow density without road pricing, they assumed inelastic travel demand ($\beta = 0$ in Eq. (2)). We relax the assumption to elastic travel demand.

First, we derive the traffic flow density in the east–west direction. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be origin and destination of trips, respectively. The traffic passes through the infinitesimal segment between two points $(x, y)$ and $(x, y + \mathrm{d}y)$ in the easterly direction if

$$P_1(x_1, y_1) \in \{(x_1, y_1) \mid 0 \leq x_1 \leq x, y \leq y_1 \leq y + \mathrm{d}y\}, \tag{5}$$
$$P_2(x_2, y_2) \in \{(x_2, y_2) \mid x \leq x_2 \leq a_1, 0 \leq y_2 \leq a_2\}, \tag{6}$$

as shown in Fig. 2a, or

$$P_1(x_1, y_1) \in \{(x_1, y_1) \mid 0 \leq x_1 \leq x, 0 \leq y_1 \leq a_2\}, \tag{7}$$
$$P_2(x_2, y_2) \in \{(x_2, y_2) \mid x \leq x_2 \leq a_1, y \leq y_2 \leq y + \mathrm{d}y\}, \tag{8}$$

as shown in Fig. 2b. Since the amount of traffic flow passing through the segment in the westerly direction is the same as that in the easterly direction, the amount of traffic flow passing through the segment in the east–west direction is

$$\begin{aligned} f_x \mathrm{d}y = & \int_0^y \int_x^{a_1} \int_y^{y+\mathrm{d}y} \int_0^x D_0 \exp\{-\alpha\beta(x_2 - x_1 + y_1 - y_2)\} \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \\ & + \int_y^{a_2} \int_x^{a_1} \int_y^{y+\mathrm{d}y} \int_0^x D_0 \exp\{-\alpha\beta(x_2 - x_1 + y_2 - y_1)\} \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \\ & + \int_y^{y+\mathrm{d}y} \int_x^{a_1} \int_0^y \int_0^x D_0 \exp\{-\alpha\beta(x_2 - x_1 + y_2 - y_1)\} \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \end{aligned}$$

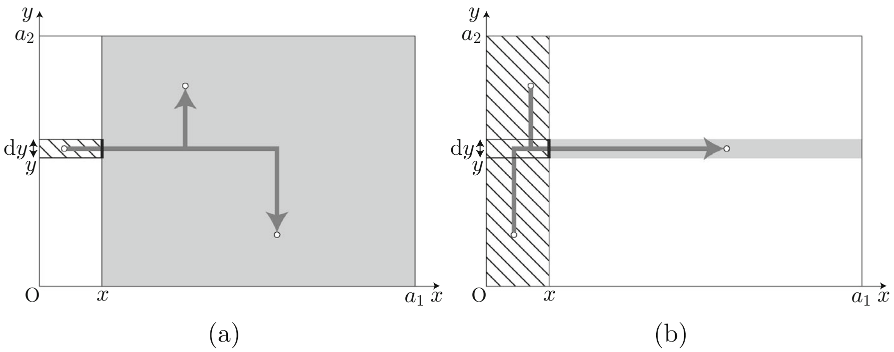


**Fig. 2** Traffic flow in the east–west direction

$$+\int_{y}^{y+\mathrm{d}y}\int_{x}^{a_1}\int_{y}^{a_2}\int_{0}^{x} D_0\exp\{-\alpha\beta(x_2-x_1+y_1-y_2)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2. \tag{9}$$

Letting $\mathrm{d}y \to +0$ gives the traffic flow density in the east–west direction

$$\begin{aligned} f_x = &\int_0^y\int_x^{a_1}\int_0^x D_0\exp\{-\alpha\beta(x_2-x_1+y-y_2)\}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2\\ &+\int_y^{a_2}\int_x^{a_1}\int_0^x D_0\exp\{-\alpha\beta(x_2-x_1+y_2-y)\}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2\\ &+\int_x^{a_1}\int_0^y\int_0^x D_0\exp\{-\alpha\beta(x_2-x_1+y-y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\\ &+\int_x^{a_1}\int_y^{a_2}\int_0^x D_0\exp\{-\alpha\beta(x_2-x_1+y_1-y)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\\ = &\frac{2D_0}{\alpha^3\beta^3}\left(1-\mathrm{e}^{\alpha\beta x}\right)\left(\mathrm{e}^{\alpha\beta a_1}-\mathrm{e}^{\alpha\beta x}\right)\left(\mathrm{e}^{\alpha\beta a_2}+\mathrm{e}^{2\alpha\beta y}-2\mathrm{e}^{\alpha\beta(a_2+y)}\right)\mathrm{e}^{-\alpha\beta(a_1+a_2+x+y)}. \end{aligned} \tag{10}$$

Next, we derive the traffic flow density in the north–south direction. The traffic passes through the infinitesimal segment between two points $(x, y)$ and $(x+\mathrm{d}x, y)$ in the northerly direction if

$$P_1(x_1, y_1) \in \{(x_1, y_1) \mid x \le x_1 \le x+\mathrm{d}x, 0 \le y_1 \le y\}, \tag{11}$$

$$P_2(x_2, y_2) \in \{(x_2, y_2) \mid 0 \le x_2 \le a_1, y \le y_2 \le a_2\}, \tag{12}$$

as shown in Fig. 3a, or

$$P_1(x_1, y_1) \in \{(x_1, y_1) \mid 0 \le x_1 \le a_1, 0 \le y_1 \le y\}, \tag{13}$$

$$P_2(x_2, y_2) \in \{(x_2, y_2) \mid x \le x_2 \le x+\mathrm{d}x, y \le y_2 \le a_2\}, \tag{14}$$

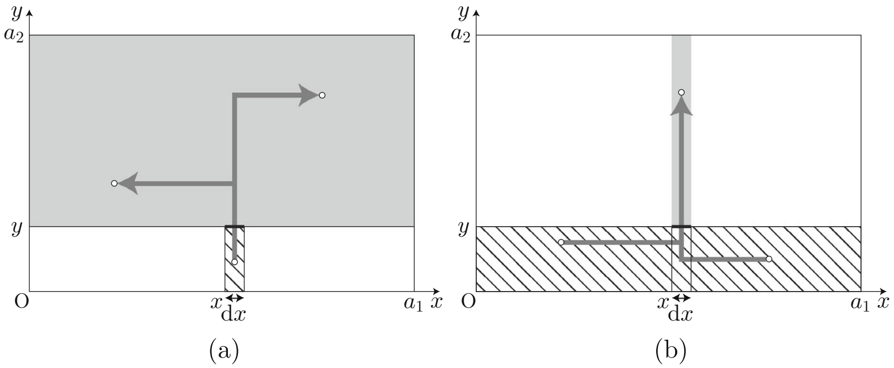


**Fig. 3** Traffic flow in the north–south direction

as shown in Fig. 3b. Since the amount of traffic flow passing through the segment in the southerly direction is the same as that in the northerly direction, the amount of traffic flow passing through the segment in the north–south direction is

$$\begin{aligned} f_y \mathrm{d}x = &\int_y^{a_2}\int_0^x\int_0^y\int_x^{x+\mathrm{d}x} D_0 \exp\{-\alpha\beta(x_1 - x_2 + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_x^{a_1}\int_0^y\int_x^{x+\mathrm{d}x} D_0 \exp\{-\alpha\beta(x_2 - x_1 + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_x^{x+\mathrm{d}x}\int_0^y\int_0^x D_0 \exp\{-\alpha\beta(x_2 - x_1 + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_x^{x+\mathrm{d}x}\int_0^y\int_x^{a_1} D_0 \exp\{-\alpha\beta(x_1 - x_2 + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2. \end{aligned} \tag{15}$$

Letting $\mathrm{d}x \to +0$ gives the traffic flow density in the north–south direction

$$\begin{aligned} f_y = &\int_y^{a_2}\int_0^x\int_0^y D_0 \exp\{-\alpha\beta(x - x_2 + y_2 - y_1)\}\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_x^{a_1}\int_0^y D_0 \exp\{-\alpha\beta(x_2 - x + y_2 - y_1)\}\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_0^y\int_0^x D_0 \exp\{-\alpha\beta(x - x_1 + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_y^{a_2}\int_0^y\int_x^{a_1} D_0 \exp\{-\alpha\beta(x_1 - x + y_2 - y_1)\}\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2 \\ = &\frac{2D_0}{\alpha^3\beta^3}\left(1 - \mathrm{e}^{\alpha\beta y}\right)\left(\mathrm{e}^{\alpha\beta a_2} - \mathrm{e}^{\alpha\beta y}\right)\left(\mathrm{e}^{\alpha\beta a_1} + \mathrm{e}^{2\alpha\beta x} - 2\mathrm{e}^{\alpha\beta(a_1+x)}\right)\mathrm{e}^{-\alpha\beta(a_1+a_2+x+y)}. \end{aligned} \tag{16}$$

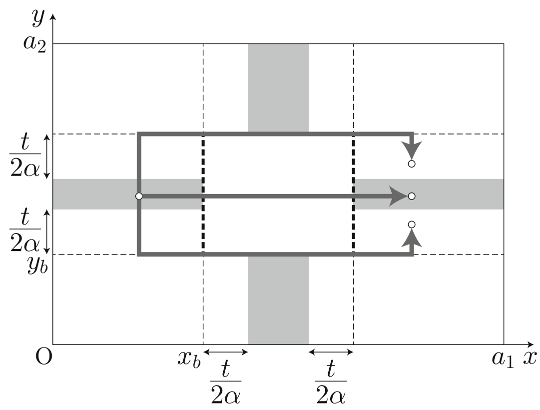


**Fig. 4** Detour around the toll area

## 4 Traffic flow density in road pricing

In this section, we derive the traffic flow density in road pricing. We then examine how the toll level, the size and shape of the toll area, and the shape of the city affect the traffic flow density. To see the effect of road pricing on the traffic flow inside the toll area, we focus on the traffic flow density inside the toll area.

Road pricing can affect not only travel demand but also travel routes, reducing the amount of traffic flow passing through the toll area. The traveler passes through the toll area only if the travel cost of passing through the toll area is smaller than that of making a detour around the toll area. Miyagawa [21] showed that the traveler passes through the toll area if both origin and destination are inside the gray regions in Fig. 4. For example, the travel cost between origin

$$P_1(x_1, y_1) \in \{(x_1, y_1) \mid 0 \le x_1 \le x_b, y_b \le y_1 \le y_2\} \tag{17}$$

and destination

$$P_2(x_2, y_2) \in \{(x_2, y_2) \mid x_b + b_1 \le x_2 \le a_1, y_b \le y_2 \le y_b + b_2\} \tag{18}$$

is given by

$$C = \begin{cases} \alpha(x_2 - x_1 + y_2 - y_1) + t, & y_1 > y_b + \frac{t}{2\alpha}, y_2 \le y_b + b_2 - \frac{t}{2\alpha}, \\ \alpha(x_2 - x_1 + y_1 + y_2 - 2y_b), & y_1 \le y_b + \frac{t}{2\alpha}, y_1 + y_2 \le 2y_b + b_2, \\ \alpha(x_2 - x_1 - y_1 - y_2 + 2y_b + 2b_2), & y_2 > y_b + b_2 - \frac{t}{2\alpha}, y_1 + y_2 > 2y_b + b_2. \end{cases} \tag{19}$$

The traveler then passes through the toll area if $y_1 > y_b + t/(2\alpha)$, $y_2 \le y_b + b_2 - t/(2\alpha)$.

The traffic flow density in the east–west direction is obtained by considering the traffic passing through the infinitesimal segment between two points $(x, y)$ and $(x, y + \mathrm{d}y)$, as shown in Fig. 5. If $y_b \le y \le y_b + t/(2\alpha)$ or $y_b + b_2 - t/(2\alpha) \le y \le y_b + b_2$,

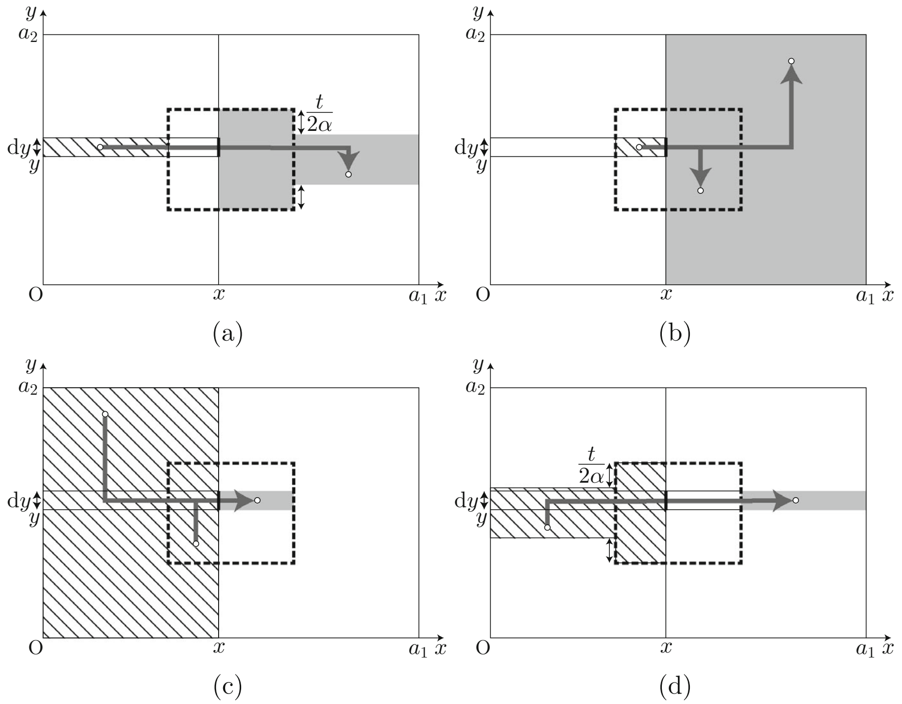


**Fig. 5** Traffic flow in the east–west direction

travelers whose origin and destination are outside the toll area do not pass through the segment. The traffic flow density in the east–west direction is given by

$$f_x = f_x^{\mathrm{a}} + f_x^{\mathrm{b}} + f_x^{\mathrm{c}} + f_x^{\mathrm{d}}, \tag{20}$$

where

$$\begin{aligned} f_x^{\mathrm{a}} = &\int_{y_b}^{y} \int_{x}^{x_b+b_1} \int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2 - x_1 + y - y_2) + t\}]\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_2 \\ &+ \int_{y}^{y_b+b_2} \int_{x}^{x_b+b_1} \int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2 - x_1 + y_2 - y) + t\}]\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_2, \end{aligned} \tag{21}$$

$$\begin{aligned} f_x^{\mathrm{b}} = &\int_{0}^{y} \int_{x}^{a_1} \int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2 - x_1 + y - y_2) + t\}]\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_2 \\ &+ \int_{y}^{a_2} \int_{x}^{a_1} \int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2 - x_1 + y_2 - y) + t\}]\, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_2, \end{aligned} \tag{22}$$

$$f_x^{\mathrm{c}} = \int_{x}^{x_b+b_1} \int_{0}^{y} \int_{0}^{x} D_0 \exp[-\beta\{\alpha(x_2 - x_1 + y - y_1) + t\}]\, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2$$

$$+\int_{x}^{x_b+b_1}\int_{y}^{a_2}\int_{0}^{x} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_1-y)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2, \quad (23)$$

$$f_x^{\mathrm{d}} = \int_{x_b+b_1}^{a_1}\int_{y_b}^{y}\int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2-x_1+y-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2$$
$$+\int_{x_b+b_1}^{a_1}\int_{y}^{y_b+b_2}\int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_1-y)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2. \quad (24)$$

Note that $f_x^{\mathrm{a}}$, $f_x^{\mathrm{b}}$, $f_x^{\mathrm{c}}$, $f_x^{\mathrm{d}}$ correspond to the four cases shown in Fig. 5. If $y_b+t/(2\alpha) < y < y_b+b_2-t/(2\alpha)$, travelers whose origin and destination are outside the toll area can pass through the segment, and $f_x^{\mathrm{a}}$ and $f_x^{\mathrm{d}}$ are replaced by

$$f_x^{\mathrm{a}} = \int_{y_b}^{y}\int_{x}^{x_b+b_1}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y-y_2)+t\}]\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2$$
$$+\int_{y}^{y_b+b_2}\int_{x}^{x_b+b_1}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_2-y)+t\}]\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2$$
$$+\int_{y_b+t/(2\alpha)}^{y}\int_{x_b+b_1}^{a_1}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y-y_2)+t\}]\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2$$
$$+\int_{y}^{y_b+b_2-t/(2\alpha)}\int_{x_b+b_1}^{a_1}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_2-y)+t\}]\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_2, \quad (25)$$

$$f_x^{\mathrm{d}} = \int_{x_b+b_1}^{a_1}\int_{y_b}^{y}\int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2-x_1+y-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2$$
$$+\int_{x_b+b_1}^{a_1}\int_{y}^{y_b+b_2}\int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_1-y)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2$$
$$+\int_{x_b+b_1}^{a_1}\int_{y_b+t/(2\alpha)}^{y}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2$$
$$+\int_{x_b+b_1}^{a_1}\int_{y}^{y_b+b_2-t/(2\alpha)}\int_{0}^{x_b} D_0 \exp[-\beta\{\alpha(x_2-x_1+y_1-y)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2. \quad (26)$$

The traffic flow density in the north–south direction is obtained by considering the traffic passing through the infinitesimal segment between two points $(x, y)$ and $(x+\mathrm{d}x, y)$, as shown in Fig. 6. If $x_b \le x \le x_b + t/(2\alpha)$ or $x_b + b_1 - t/(2\alpha) \le x \le x_b + b_1$, travelers whose origin and destination are outside the toll area do not pass through the segment. The traffic flow density in the north–south direction is given by

$$f_y = f_y^{\mathrm{a}} + f_y^{\mathrm{b}} + f_y^{\mathrm{c}} + f_y^{\mathrm{d}}, \quad (27)$$

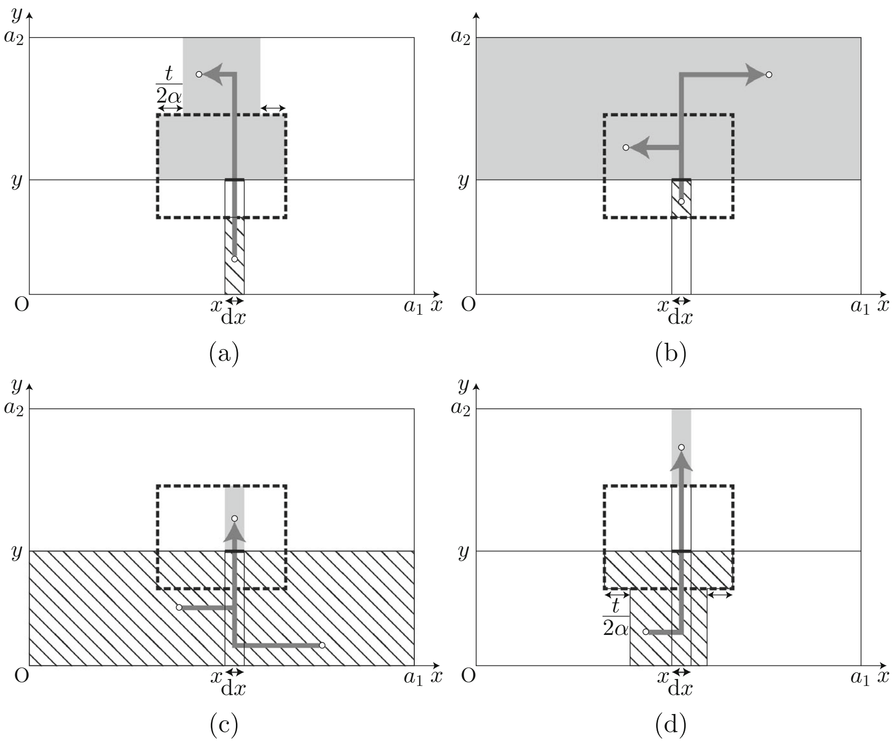


**Fig. 6** Traffic flow in the north–south direction

where

$$f_y^{\mathrm{a}} = \int_y^{y_b+b_2} \int_{x_b}^{x} \int_0^{y_b} D_0 \exp[-\beta\{\alpha(x - x_2 + y_2 - y_1)\} + t\}] \, \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 + \int_y^{y_b+b_2} \int_x^{x_b+b_1} \int_0^{y_b} D_0 \exp[-\beta\{\alpha(x_2 - x + y_2 - y_1) + t\}] \, \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2, \tag{28}$$

$$f_y^{\mathrm{b}} = \int_y^{a_2} \int_0^{x} \int_{y_b}^{y} D_0 \exp[-\beta\{\alpha(x - x_2 + y_2 - y_1)\} + t\}] \, \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 + \int_y^{a_2} \int_x^{a_1} \int_{y_b}^{y} D_0 \exp[-\beta\{\alpha(x_2 - x + y_2 - y_1) + t\}] \, \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2, \tag{29}$$

$$f_y^{\mathrm{c}} = \int_y^{y_b+b_2} \int_0^{y} \int_0^{x} D_0 \exp[-\beta\{\alpha(x - x_1 + y_2 - y_1) + t\}] \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}y_2 + \int_y^{y_b+b_2} \int_0^{y} \int_x^{a_1} D_0 \exp[-\beta\{\alpha(x_1 - x + y_2 - y_1) + t\}] \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}y_2, \tag{30}$$

$$f_y^{\mathrm{d}} = \int_{y_b+b_2}^{a_2} \int_{y_b}^{y} \int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x - x_1 + y_2 - y_1) + t\}] \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}y_2$$

$$+\int_{y_b+b_2}^{a_2}\int_{y_b}^{y}\int_{x}^{x_b+b_1} D_0 \exp[-\beta\{\alpha(x_1-x+y_2-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2. \tag{31}$$

Note that $f_y^{\mathrm{a}}$, $f_y^{\mathrm{b}}$, $f_y^{\mathrm{c}}$, $f_y^{\mathrm{d}}$ correspond to the four cases shown in Fig. 6. If $x_b+t/(2\alpha) < x < x_b+b_1-t/(2\alpha)$, travelers whose origin and destination are outside the toll area can pass through the segment, and $f_y^{\mathrm{a}}$ and $f_y^{\mathrm{d}}$ are replaced by

$$\begin{aligned} f_y^{\mathrm{a}} &= \int_{y}^{y_b+b_2}\int_{x_b}^{x}\int_{0}^{y_b} D_0 \exp[-\beta\{\alpha(x-x_2+y_2-y_1)\}+t\}]\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad+\int_{y}^{y_b+b_2}\int_{x}^{x_b+b_1}\int_{0}^{y_b} D_0 \exp[-\beta\{\alpha(x_2-x+y_2-y_1)+t\}]\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad+\int_{y_b+b_2}^{a_2}\int_{x_b+t/(2\alpha)}^{x}\int_{0}^{y_b} D_0 \exp[-\beta\{\alpha(x-x_2+y_2-y_1)\}+t\}]\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad+\int_{y_b+b_2}^{a_2}\int_{x}^{x_b+b_1-t/(2\alpha)}\int_{0}^{y_b} D_0 \exp[-\beta\{\alpha(x_2-x+y_2-y_1)+t\}]\,\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2, \end{aligned} \tag{32}$$

$$\begin{aligned} f_y^{\mathrm{d}} &= \int_{y_b+b_2}^{a_2}\int_{y_b}^{y}\int_{x_b}^{x} D_0 \exp[-\beta\{\alpha(x-x_1+y_2-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad+\int_{y_b+b_2}^{a_2}\int_{y_b}^{y}\int_{x}^{x_b+b_1} D_0 \exp[-\beta\{\alpha(x_1-x+y_2-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad+\int_{y_b+b_2}^{a_2}\int_{0}^{y_b}\int_{x_b+t/(2\alpha)}^{x} D_0 \exp[-\beta\{\alpha(x-x_1+y_2-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2 \\ &\quad+\int_{y_b+b_2}^{a_2}\int_{0}^{y_b}\int_{x}^{x_b+b_1-t/(2\alpha)} D_0 \exp[-\beta\{\alpha(x_1-x+y_2-y_1)+t\}]\,\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}y_2. \end{aligned} \tag{33}$$

The traffic flow densities $f = f_x + f_y$ inside the toll area for the three toll levels ($t = 0, 0.1, 0.5$) are shown in Fig. 7, where $a_1 = a_2 = 1, b_1 = b_2 = 0.6, D_0 = 1, \alpha = 1, \beta = 1$. As the toll level increases, the traffic flow density inside the toll area decreases. Note that the traffic flow density jumps at $x = x_b+t/(2\alpha)$, $x_b+b_1-t/(2\alpha)$ and $y = y_b+t/(2\alpha)$, $y_b+b_2-t/(2\alpha)$. This is because for $x_b+t/(2\alpha) < x < x_b+b_1-t/(2\alpha)$ and $y_b+t/(2\alpha) < y < y_b+b_2-t/(2\alpha)$, some travelers whose origin and destination are outside the toll area pass through the toll area, as shown in Fig. 4.

To examine the effect of the size of the toll area on the traffic flow density, we consider two sizes of the toll area: $b_1 = b_2 = 0.4$ and $b_1 = b_2 = 0.6$. The traffic flow densities inside the toll area at $y = a_2/2$ for the two sizes are shown in Fig. 8, where $a_1 = a_2 = 1, D_0 = 1, \alpha = 1, \beta = 1$. Note that if the toll area is small, the traffic flow density at the city center decreases rapidly with the toll level $t$. This is because as the size of the toll area becomes smaller, the detour distance to avoid the toll area decreases and much traffic makes a detour around the toll area. It follows that

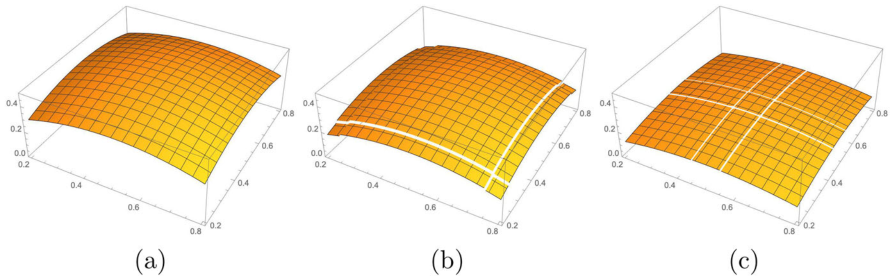


**Fig. 7** Traffic flow density inside the toll area: **a** $t = 0$; **b** $t = 0.1$; **c** $t = 0.5$

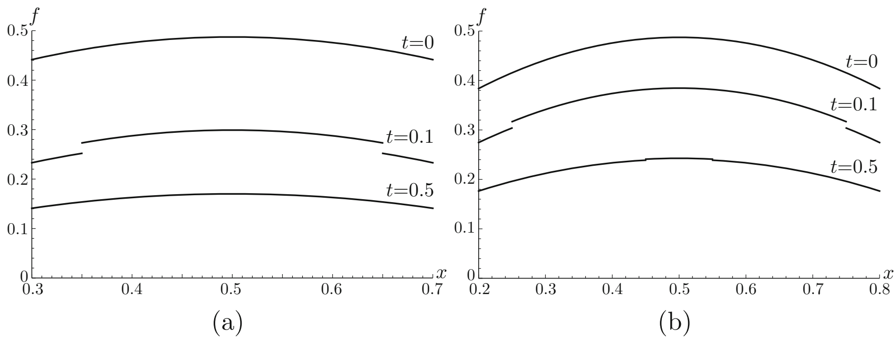


**Fig. 8** Traffic flow density inside the toll area at $y = a_2/2$: **a** $b_1 = b_2 = 0.4$; **b** $b_1 = b_2 = 0.6$

to reduce the amount of traffic flow at the city center, the toll area should be small and the toll level should be high.

To examine the effect of the shape of the toll area on the traffic flow density, we consider two shapes of the toll area: $b_1/b_2 = 2\,(b_1 = 0.4\sqrt{2}, b_2 = 0.4/\sqrt{2})$ and $b_1/b_2 = 3\,(b_1 = 0.4\sqrt{3}, b_2 = 0.4/\sqrt{3})$. The traffic flow densities inside the toll area at $y = a_2/2$ for the two shapes are shown in Fig. 9, where $a_1 = a_2 = 1$, $D_0 = 1$, $\alpha = 1$, $\beta = 1$. The traffic flow density for $b_1/b_2 = 1\,(b_1 = b_2 = 0.4)$ is shown in Fig. 8a. Note that as the aspect ratio of the toll area $b_1/b_2$ becomes larger, the traffic flow density at the city center increases. Thus, reducing the amount of traffic flow at the city center is easier when the toll area is a square.

To examine the shape of the city on the traffic flow density, we consider two shapes of the city: $a_1/a_2 = 2\,(a_1 = \sqrt{2}, a_2 = 1/\sqrt{2})$ and $a_1/a_2 = 3\,(a_1 = \sqrt{3}, a_2 = 1/\sqrt{3})$. The traffic flow densities inside the toll area at $y = a_2/2$ for the two shapes are shown in Fig. 10, where $b_1 = b_2 = 0.4$, $D_0 = 1$, $\alpha = 1$, $\beta = 1$. The traffic flow density for $a_1/a_2 = 1\,(a_1 = a_2 = 1)$ is shown in Fig. 8a. Note that the shape of the city has less impact on the traffic flow density than the shape of the toll area.

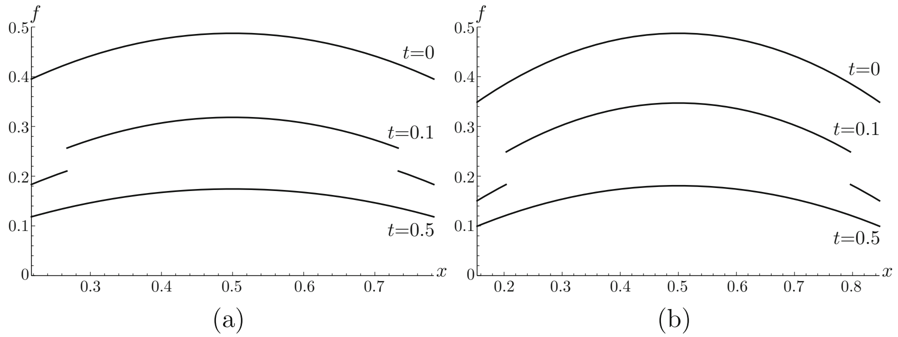


**Fig. 9** Traffic flow density inside the toll area at $y = a_2/2$: **a** $b_1/b_2 = 2$; **b** $b_1/b_2 = 3$

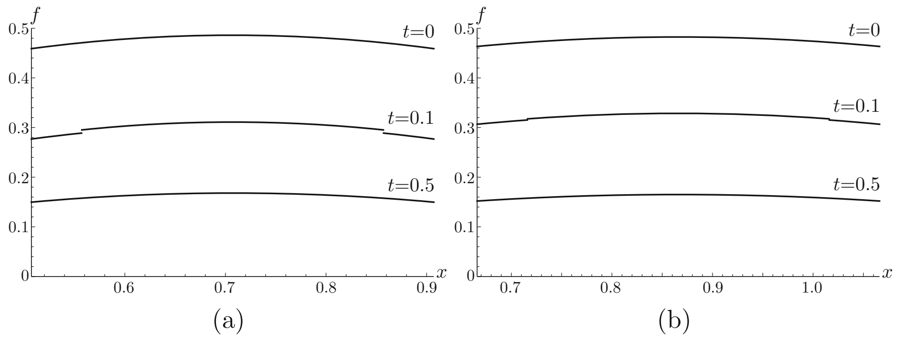


**Fig. 10** Traffic flow density inside the toll area at $y = a_2/2$: **a** $a_1/a_2 = 2$; **b** $a_1/a_2 = 3$

## 5 Conclusions

This paper has developed a continuous approximation model for analyzing the spatial distribution of traffic flow in road pricing. The traffic flow density has been derived for a rectangular city with a grid network. The model provides a fundamental understanding of the effect of road pricing on the spatial distribution of traffic flow.

The traffic flow density is useful for designing road pricing systems. The analytical expression for the traffic flow density demonstrates how the toll level, the size and shape of the toll area, and the shape of the city affect the spatial distribution of traffic flow. As the size of the toll area increases, reducing the traffic flow density at the city center becomes difficult. As the aspect ratio of the toll area increases, the traffic flow density at the city center increases. The shape of the city has less impact on the traffic flow density than the shape of the toll area. Note that finding these relationships by using discrete network models requires the computation of traffic flow for various combinations of the parameters. These relationships help planners determine the size and shape of the toll area and the toll level.

Not only area-based pricing but also other pricing schemes such as cordon-based and distance-based pricing have been studied. In particular, marginal cost pricing is the first-best pricing in that the social welfare is maximized by internalizing externalities of congestion [34]. Future research should compare these pricing schemes in terms of the spatial distribution of traffic flow.

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## Declarations

**Conflcit of interest** The author has no competing interests to declare that are relevant to the content of this article.



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